Generalized functional-integral approach to time evolution of a coupled photon system

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(Received 14 January 1997)

A generalized functional-integral approach is presented to describe the coupled photon system in a free-electron laser. By establishing the different complex time paths to two subsystems, the Wiggler field and the laser field, an effective action of the system is introduced. A general photon distribution function is obtained by evaluating the functional integral for the generating function. [S1063-651X(97)05009-5]

PACS number(s): 05.30.−d, 41.60.Cr

The problem of the signal gain of a free-electron laser has been intensively studied for many years. However, previous investigations have concentrated on discussing the small signal gain in the framework of the perturbation method [1]. As the interaction between the Wiggler field and the laser field in a free-electron laser is often very strong, the perturbation method is no longer valid. We note that the functional-integral technique can be used to study the nonequilibrium problem [2–5]. In particular, Pötz and Zhang investigated the time evolution of the electron-phonon system with this technique. However, their approach could not be used to study the coupled photon system in a free-electron laser, as it is still a perturbative one.

The focus of this work is to develop a convenient functional-integral approach that can be used to calculate the photon distribution function for any signal gain. By establishing different complex time paths to two subsystems, the Wiggler field and the laser field, an effective action of the coupled photon system can be introduced. Therefore the exact evaluation of the generating function will become possible, and a nonperturbative result will be obtained. In the remainder of the paper, we shall introduce an effective action for the coupled photon system, and derive the photon distribution function by evaluating the functional integral.

The coupled photon system in a free-electron laser consists of two subsystems, the Wiggler field and the laser field. Each one is assumed to have a complex time coordinate defined as

$$\tilde{t}=\tau+i\,\frac{t}{\hbar} \tag{1}$$

where the domains for both the real part τ and the imaginary part t are $0\sim\beta$ and $0\sim T$, respectively, where β is the reciprocal temperature and T the duration time. The definition of the complex time is the same as that by Pötz and Zhang [5].

One may construct the Hamiltonian for two subsystems with complex time $\tilde{t}_l$ for the laser field and $\tilde{t}_w$ for the Wiggler field,

$$H(\tilde{t}_l,\tilde{t}_w)=H_l^0(\tilde{t}_l)+H_w^0(\tilde{t}_w)+V(\tilde{t}_l,\tilde{t}_w), \tag{2}$$

where $H_\sigma^0(\tilde{t}_\sigma)$ $(\sigma=l,w)$ is the Hamiltonian of the free subsystem, and $V(\tilde{t}_l,\tilde{t}_w)$ is the interaction between two subsystems. By integrating the Hamiltonian with respect to $\tilde{t}_l$ and $\tilde{t}_w$ along the Pötz-Zhang complex time path shown in Fig. 1, we have

$$\int_{\tilde{t}_f}^{\tilde{t}_f}\int_{\tilde{t}_i}^{\tilde{t}_f} H_l^0(\tilde{t}_l)\,d\tilde{t}_l\,d\tilde{t}_w+\int_{\tilde{t}_i}^{\tilde{t}_f}\int_{\tilde{t}_i}^{\tilde{t}_f} H_w^0(\tilde{t}_w)\,d\tilde{t}_l\,d\tilde{t}_w + \int_{\tilde{t}_i}^{\tilde{t}_f}\int_{\tilde{t}_i}^{\tilde{t}_f} V(\tilde{t}_l,\tilde{t}_w)\,d\tilde{t}_l\,d\tilde{t}_w=\beta S_{l,w}, \tag{3}$$

where $S_{l,w}$ is regarded as an effective action of the system for the reason of its dimension.

The ensemble average for an observable is expressed as

$$\langle A\rangle=\frac{1}{Z}\,Tr(A\rho), \tag{4}$$

where $Z=Tr(\rho)$ is the grand partition function of the system, and ρ the density operator,

$$\begin{aligned}\rho&=\exp(-S_{l,w})\\&=\exp\left[-\int_{\tilde{t}_i}^{\tilde{t}_f}H_l^0(\tilde{t}_l)\,d\tilde{t}_l-\int_{\tilde{t}_i}^{\tilde{t}_f}H_w^0(\tilde{t}_w)\,d\tilde{t}_w\right.\\&\quad\left.-\frac{1}{\beta}\int_{\tilde{t}_i}^{\tilde{t}_f}\int_{\tilde{t}_i}^{\tilde{t}_f}V(\tilde{t}_l,\tilde{t}_w)\,d\tilde{t}_l\,d\tilde{t}_w\right].\end{aligned} \tag{5}$$

The problem of the photon excitation in a free-electron laser can be reduced to a nonrelativistic one by properly choosing a frame where the frequency of the radiation field equals the frequency of the magnetic-field periods [1]. The laser radiation field and the wiggler pseudoradiation field in

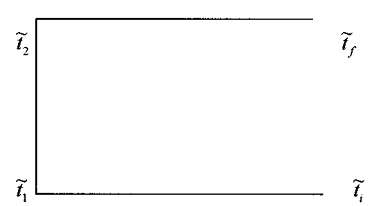

FIG. 1. The integration path in the complex time plane (Pötz and Zhang's path) with the parameters $\tilde{t}_i=0+iT$, $\tilde{t}_1=0+i0$, $\tilde{t}_2=\beta+i0$, and $\tilde{t}_f=\beta+iT$.

this frame have the same frequency ω and opposite wave vectors of equal magnitude k. The Hamiltonian of the system $H(\tilde{t}_l,\tilde{t}_w)$ is written in terms of the creation and annihilation operators a_σ^* and a_σ $(\sigma=l,w)$,

$$H(\tilde{t}_l,\tilde{t}_w)=\hbar\omega[a_l^*(\tilde{t}_l)a_l(\tilde{t}_l)+a_w^*(\tilde{t}_w)a_w(\tilde{t}_w)]+\hbar\Lambda[a_l(\tilde{t}_l)a_w^*(\tilde{t}_w)e^{-ik\nu_z(\tilde{t}_l+\tilde{t}_w)}+a_w(\tilde{t}_w)a_l^*(\tilde{t}_l)e^{ik\nu_z(\tilde{t}_l+\tilde{t}_w)}], \tag{6}$$

where

$$\Lambda=\frac{2\pi c r_0}{kV}. \tag{7}$$

r_0 is the classical radius of the electron, k the wave vector of the photon, V the volume of the interacting region, and ν_z the electron velocity.

Equation (4) in the generating functional approach can be written as [5]

$$\langle A\rangle=\frac{A(\partial/\partial J,\partial/\partial J^*)g(J,J^*)}{g(J,J^*)}\Bigg|_{\substack{J=0\\ J^*=0}}, \tag{8}$$

where $g(J,J^*)$ is the generating function. The complex time path in the functional integral is divided into M fractions, and the interval of the sth fraction is

$$\varepsilon_s=\tilde{t}_{s+1}-\tilde{t}_s. \tag{9}$$

According to the complex time lattices, the discrete action of the system can be expressed as

$$S_{l,w}=\sum_n(a_{\ln}^*a_{\ln}-a_{\ln}^*a_{\ln-1})+\sum_m(a_{wm}^*a_{wm}-a_{wm}^*a_{wm-1})+\hbar\omega\left(\sum_n\varepsilon_n a_{\ln}^*a_{\ln}+\sum_m\varepsilon_m a_{wm}^*a_{wm-1}\right)+\sum_n\sum_m\chi_{n-1}\chi_m\varepsilon_{n-1}\varepsilon_m a_{\ln-1}a_{wm}^*+\sum_n\sum_m\chi_n^*\chi_{m-1}^*\varepsilon_n\varepsilon_{m-1}\alpha_{wm-1}a_{\ln}^*+J^*a_{M-1}+a_M^*J, \tag{10}$$

where

$$\chi_n=\left(\frac{\hbar\Lambda}{\beta}\right)^{1/2}e^{-ik\nu_z\tilde{t}_n}, \tag{11}$$

and $\{J^*,J\}$ are the external sources.

The generating function is

$$g(J,J^*)=\prod_n\prod_m\frac{da_{\ln}^*da_{\ln}}{2\pi i}\frac{da_{wm}^*da_{wm}}{2\pi i}e^{-S_{l,w}}. \tag{12}$$

The integration over the photon field in Eq. (12) can be carried out explicitly

$$g(J,J^*)=\frac{1}{(1-e^{-\beta\hbar\omega})[1-e^{-\Sigma_{s=1}^M\varepsilon_s(\hbar\omega-F_3|\chi|^2)}]}\times\exp\left\{J^*\left[\frac{1}{e^{\beta\hbar\omega}-1}+\frac{F_1F_2\sum_{n=1}^{M-1}\varepsilon_n\chi_n e^{-\Sigma_{s=1}^n\varepsilon_s(\hbar\omega-F_3|\chi_s|^2)}\sum_{m=1}^{M-1}\varepsilon_m\chi_m^*e^{-\Sigma_{s=m+1}^M\varepsilon_s(\hbar\omega-F_3|\chi_s|^2)}}{1-e^{-\sum_{s=1}^M\varepsilon_s(\hbar\omega-F_3|\chi_s|^2)}}\right]J\right\}, \tag{13}$$

where

$$F_1=\frac{1}{1-e^{-\beta\hbar\omega}}\sum_{n=1}^{M-1}\varepsilon_n\chi_n e^{-\Sigma_{s=n+1}^M\hbar\omega\varepsilon_s}, \tag{14}$$

$$F_2=\frac{1}{1-e^{-\beta\hbar\omega}}\sum_{n=1}^{M-1}\varepsilon_n\chi_n^*e^{-\Sigma_{s=1}^n\hbar\omega\varepsilon_s}, \tag{15}$$

and

$$F_3=(1-e^{-\beta\hbar\omega})F_1F_2. \tag{16}$$

The distribution function of the laser photons is

$$N=\frac{\dfrac{\delta^2}{\delta J\,\delta J^*}g(J,J^*)}{g(J,J^*)}\Bigg|_{\substack{J=0\\ J^*=0}}. \tag{17}$$

By differentiating the $g(J,J^*)$ with respect to J and J^* in Eq. (17), we find

$$N=\frac{1}{e^{\beta\hbar\omega}-1}+\frac{F_1F_2\sum_{n=1}^{M-1}\varepsilon_n\chi_n e^{-\Sigma_{s=1}^{n}\varepsilon_s[\hbar\omega-F_3(\hbar\Lambda/\beta)]}\sum_{m=1}^{M}\varepsilon_m\chi_m e^{-\Sigma_{s=m+1}^{M}\varepsilon_s[\hbar\omega-F_3(\hbar\Lambda/\beta)]}}{1-e^{-\beta[\hbar\omega-F_3(\hbar\Lambda/\beta)]}}. \tag{18}$$

In the limit of $M\rightarrow\infty$, the discrete summations in Eq. (18) are replaced by integration, and we find

$$N=\frac{1}{e^{\beta\hbar\omega}-1}+[1-e^{-\beta[\hbar\omega-(F_3\hbar\Lambda/\beta)]}]\frac{\Lambda^2}{\hbar^2\beta^2}\left[\frac{1}{\omega^2}+\frac{k\nu_z}{\omega}\times\frac{4\sin^2\frac{1}{2}(k\nu_z-\omega)T}{(k\nu_z-\omega)^2}\right]\left[\frac{1}{[\omega-(F_3\Lambda/\beta)]^2}-\frac{k\nu_z}{[\omega-(F_3\Lambda/\beta)]}\frac{4\sin^2\frac{1}{2}[k\nu_z+\omega-(F_3\Lambda/\beta)]T}{[k\nu_z+\omega-(F_3\Lambda/\beta)]^2}\right], \tag{19}$$

where

$$F_3=(1-e^{-\beta\hbar\omega})\frac{\Lambda}{\hbar}\left[\frac{1}{\omega^2}+\frac{k\nu_z}{\omega}\frac{4\sin^2\frac{1}{2}(k\nu_z-\omega)T}{(k\nu_z-\omega)^2}\right]. \tag{20}$$

Thus we have derived a general expression for the photon distribution. The results from the perturbative approach can be obtained from Eq. (19) in the weak-coupling limit $\Lambda \ll 1$, while the photon distribution function for an equilibrium state

$$N=\frac{1}{e^{\beta\hbar\omega}-1} \tag{21}$$

is a consequence of Eq. (19) with $\Lambda=0$.

In conclusion, the coupled photon system is divided into two subsystems with different complex time coordinates. An effective action is constructed based on the requirement of the dimension. The complex time path in the functional integral is the same as the Pötz and Zhang's path. This generalized functional-integral technique is applied to the coupled photon system of a free-electron laser, and a new expression of the photon distribution function is obtained.

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